AN ANALOGUE OF THE LÉVY-CRAMÉR THEOREM FOR MULTI-DIMENSIONAL RAYLEIGH DISTRIBUTIONS

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ABSTRACT. In the present paper we prove that every k-dimensional Cartesian product of Kingman convolutions can be embedded into a k-dimensional symmetric convolution (k=1, 2, ...) and obtain an analogue of the Cramér-Lévy theorem for multi-dimensional Rayleigh distributions. A new and more general class of multi-dimensional Rayleigh distributions and associated higher dimensional Bessel processes are introduced and studied. This class of processes inherits the well-known characteristics of Brownian motions: They are independent stationary "increments" processes with continuous sample paths.

Keywords and phrases: Cartesian products of Kingman convolutions; Rayleigh distributions; radial characteristic functions; k-symmetric Brownian motions; k-symmetric Lévy processes.

AMS2000 subject classification: 60B07, 60B11, 60B15, 60K99.

1. Introduction, Notations and Prelimilaries

In probability theory and statistics, the **Rayleigh distribution** is a continuous probability distribution which is widely used to model events that occur in different fields such as medicine, social and natural sciences. A multivariate Rayleigh distribution is the probability distribution of a vector of norms of random Gaussian vectors. The purpose of this paper, is to introduce and study the fractional indexes multivariate Rayleigh distributions via the Cartesian product of Kingman convolutions and, in particular, to prove an analogue of the Lévy-Cramér theorem for multivariate Rayleigh distributions. We begin with a brief review of the Kingman convolution $*_{1,\delta}$ as follows. Let $\mathcal{P}(\mathbb{R}^+)$ denotes the set of all probability measures (p.m.'s) on the positive half-line $\mathbb{R}^+$. Put, for each continuous bounded function f on $\mathbb{R}^+$,

$$(1)\quad \int_0^\infty f(x)\mu *_{1,\delta} \nu(dx) = \frac{\Gamma(s+1)}{\sqrt{\pi}\Gamma(s+\frac{1}{2})} \int_0^\infty \int_0^\infty \int_{-1}^1 f((x^2+2uxy+y^2)^{1/2})(1-u^2)^{s-1/2}\mu(dx)\nu(dy)du,$$

where μ and $\nu \in \mathcal{P}(\mathbb{R}^+)$ and $\delta = 2(s+1) \geq 1$ (cf. Kingman [7] and Urbanik [17]). The convolution algebra $(\mathcal{P}, *_{1,\delta})$ is the most important example of Urbanik convolution algebras (cf Urbanik [17]). In language of the Urbanik convolution algebras, the *characteristic measure*, say σ_s, of the Kingman convolution has the Rayleigh density

$$(2)\qquad d\sigma_s(y) = \frac{2(s+1)^{s+1}}{\Gamma(s+1)} y^{2s+1} \exp(-(s+1)y^2)dy$$

Date: June 30, 2009.

with the characteristic exponent $\varkappa = 2$ and the kernel Λ_s

$$\Lambda_s(x) = \Gamma(s+1)J_s(x)/(1/2x)^s, \tag{3}$$

where $J_s(x)$ denotes the Bessel function of the first kind,

$$J_s(x) := \Sigma_{k=0}^{\infty}\frac{(-1)^k(x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}. \tag{4}$$

It is known (cf. Kingman [7], Theorem 1), that the kernel Λ_s itself is an ordinary characteristic function (ch.f.) of a symmetric p.m., say F_s, defined on the interval [-1,1]. Thus, if θ_s denotes a random variable (r.v.) with distribution F_s then for each $t \in \mathbb{R}^+$,

$$\Lambda_s(t) = E\exp(it\theta_s) = \int_{-1}^{1}\exp(itx)dF_s(x). \tag{5}$$

Suppose that X is a nonnegative r.v. with distribution $\mu \in \mathcal{P}$ and X is independent of θ_s. The *radial characteristic function* (rad.ch.f.) of μ, denoted by $\hat{\mu}(t)$, is defined by

$$\hat{\mu}(t) = E\exp(itX\theta_s) = \int_0^{\infty}\Lambda_s(tx)\mu(dx), \tag{6}$$

for every $t \in \mathbb{R}^+$. The characteristic measure of the Kingman convolution $*_{1,\delta}$, denoted by σ_s, has the Maxwell density function

$$\frac{d\sigma_s(x)}{dx} = \frac{2(s+1)^{s+1}}{\Gamma(s+1)}x^{2s+1}exp\{-(s+1)x^2\}, \quad (0 < x < \infty). \tag{7}$$

and the rad.ch.f.

$$\hat{\sigma}_s(t) = exp\{-t^2/4(s+1)\}. \tag{8}$$

Example 1. *The case $\delta = 1$ $(s = -\frac{1}{2})$ the Kingman convolution reduces to the symmetric convolution $*_{1,1}$ with $\Lambda_{-\frac{1}{2}}(x) = cosx$ and $\varkappa = 2$, and the characteristic measure $\sigma_{-1/2}$ has the density function*

$$\frac{d\sigma_{-\frac{1}{2}}(x)}{dx} = (\pi)^{-\frac{1}{2}}exp(-x^2/2)$$

which is the density of a symmetric Gaussian distribution induced on $\mathbb{R}^+$.

2. Cartesian product of Kingman convolutions

Denote by $\mathbb{R}^{+k}, k = 1, 2, ...$ the k-dimensional nonnegative cone of $\mathbb{R}^k$ and $\mathcal{P}(\mathbb{R}^{+k})$ the class of all p.m.'s on $\mathbb{R}^{+k}$ equipped with the weak convergence. In the sequel, we will denote the multidimensional vectors and random vectors (r.vec.'s) and their distributions by bold face letters. For each point z of any set Z let δ_z denote the Dirac measure (the unit mass) at the point z. In particular, if $\mathbf{x} = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^{k+}$, then

$$\delta_{\mathbf{x}} = \delta_{x_1} \times \delta_{x_2} \times \ldots \times \delta_{x_k}, \quad (k\ times), \tag{9}$$

where the sign " $\times$ " denotes the Cartesian product of measures. We put, for $\mathbf{x} = (x_1, \cdots, x_k)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_k) \in \mathbb{R}^{+k}$,

$$\delta_{\mathbf{x}} \bigcirc_{s,k} \delta_{\mathbf{y}} = \{\delta_{x_1} \circ_s \delta_{y_1}\} \times \{\delta_{x_2} \circ_s \delta_{y_2}\} \times \cdots \times \{\delta_{x_k} \circ_s \delta_{y_k}\}, \quad (k\ times), \tag{10}$$

here and somewhere below for the sake of simplicity we denote the Kingman convolution operation $*_{1,\delta}, \delta = 2(s+1) \geq 1$ simply by $\circ_s, s \geq -\frac{1}{2}$. Since convex combinations of p.m.'s of the form (9) are dense in $\mathcal{P}(\mathbb{R}^{+k})$ the relation (10) can be extended to arbitrary p.m.'s $\mathbf{F}$ and $\mathbf{G} \in \mathcal{P}(\mathbb{R}^{+k})$. Namely, we put

$$\mathbf{F} \bigcirc_{s,k} \mathbf{G} = \iint_{\mathbb{R}^{+k}} \delta_{\mathbf{x}} \bigcirc_{s,k} \delta_{\mathbf{y}} \mathbf{F}(d\mathbf{x})\mathbf{G}(d\mathbf{y}) \tag{11}$$

which means that for each continuous bounded function ϕ defined on $\mathbb{R}^{+k}$

$$\int_{\mathbb{R}^{+k}} \phi(\mathbf{z})\mathbf{F} \bigcirc_{s,k} \mathbf{G}(d\mathbf{z}) = \iint_{\mathbb{R}^{+k}} \{ \int_{\mathbb{R}^{+k}} \phi(\mathbf{z})\delta_{\mathbf{x}} \bigcirc_{s,k} \delta_{\mathbf{y}}(d\mathbf{z})\}\mathbf{F}(d\mathbf{x})\mathbf{G}(d\mathbf{y}). \tag{12}$$

In the sequel, the binary operation $\bigcirc_{s,k}$ will be called *the k-times Cartesian product of Kingman convolutions* and the pair $(\mathcal{P}(\mathbb{R}^{+k}), \bigcirc_{s,k})$ will be called *the k-dimensional Kingman convolution algebra*. It is easy to show that the binary operation $\bigcirc_{s,k}$ is continuous in the weak topology which together with (1) and (11) implies the following theorem.

Theorem 1. *The pair $(\mathcal{P}(\mathbb{R}^{+k}), \bigcirc_{s,k})$ is a commutative topological semigroup with $\delta_{\mathbf{0}}$ as the unit element. Moreover, the operation $\bigcirc_{s,k}$ is distributive w.r.t. convex combinations of p.m.'s in $\mathcal{P}(\mathbb{R}^{+k})$.*

For every $\mathbf{G} \in \mathcal{P}(\mathbb{R}^{+k})$ the k-dimensional rad.ch.f. $\hat{\mathbf{G}}(\mathbf{t}), \mathbf{t} = (t_1, t_2, \cdots t_k) \in \mathbb{R}^{+k}$, is defined by

$$\hat{\mathbf{G}}(\mathbf{t}) = \int_{\mathbb{R}^{+k}} \prod_{j=1}^{k} \Lambda_s(t_j x_j)\mathbf{G}(\mathbf{dx}), \tag{13}$$

where $\mathbf{x} = (x_1, x_2, \cdots x_k) \in \mathbb{R}^{+k}$. Let $\mathbf{\Theta_s} = \{\theta_{s,1}, \theta_{s,2}, \cdots, \theta_{s,k}\}$, where $\theta_{s,j}$ are independent r.v.'s with the same distribution F_s. Next, assume that $\mathbf{X} = \{X_1, X_2, ..., X_k\}$ is a k-dimensional r.vec. with distribution $\mathbf{G}$ and $\mathbf{X}$ is independent of $\mathbf{\Theta}_s$. We put

$$[\mathbf{\Theta}_s, \mathbf{X}] = \{\theta_{s,1}X_1, \theta_{s,2}X_2, ..., \theta_{s,k}X_k\}. \tag{14}$$

Then, the following formula is equivalent to (13) (cf. [13])

$$\widehat{\mathbf{G}}(\mathbf{t}) = Ee^{i<\mathbf{t},[\mathbf{\Theta}_s,\mathbf{X}]>}, \qquad \mathbf{t} \in \mathbb{R}^{+k}. \tag{15}$$

The Reader is referred to Corollary 2.1, Theorems 2.3 & 2.4 [13] for the principal properties of the above rad.ch.f. Given $s \geq -1/2$ define a map $H_{s,k} : \mathcal{P}(\mathbb{R}^{+k}) \to \mathcal{P}(\mathbb{R}^k)$ by

$$H_{s,k}(\mathbf{G}) = \int_{\mathbb{R}^{+k}} (T_{c_1}F_s) \times (T_{c_2}F_s) \times \ldots \times (T_{c_k}F_s)\mathbf{G}(d\mathbf{c}), \tag{16}$$

here and in the sequel, for a distribution $\mathbf{F}$ of a r.vec. $\mathbf{X}$ and a real number c we denote by $T_c\mathbf{F}$ the distribution of $c\mathbf{X}$. Let us denote by $\tilde{\mathcal{P}}_{s,k}(\mathbb{R}^{+k})$ the sub-class of $\mathcal{P}(\mathbb{R}^k)$ consisted of all symmetric p.m.'s defined by the right-hand side of (16). By virtue of (13)-(16) it is easy to prove the following theorem.

Theorem 2. *The set $\tilde{\mathcal{P}}_{s,k}(\mathbb{R}^{+k})$ is closed w.r.t. the weak convergence and the ordinary convolution $*$ and the following equation holds*

$$\hat{\mathbf{G}}(\mathbf{t}) = \mathcal{F}(H_{s,k}(\mathbf{G}))(\mathbf{t}) \qquad \mathbf{t} \in \mathbb{R}^{+k} \tag{17}$$

where $\mathcal{F}(\mathbf{K})$ denotes the ordinary characteristic function (Fourier transform) of a p.m. $\mathbf{K}$. Therefore, for any $\mathbf{G}$ and $\mathbf{K} \in \mathbb{R}^{+k}$

$$H_{s,k}(\mathbf{G}) * H_{s,k}(\mathbf{K}) = H_{s,k}(\mathbf{G} \bigcirc_{s,k} \mathbf{K}) \tag{18}$$

and the map $H_{s,k}$ commutes with convex combinations of distributions and scale changes $T_c, c > 0$. Moreover,

$$H_{s,k}(\Sigma_{s,k}) = N(\mathbf{0}, \mathbf{I}) \tag{19}$$

where $\Sigma_{s,k}$ denotes the k-dimensional Rayleigh distribution (Definition 3) and $N(\mathbf{0}, \mathbf{I})$ is the standard normal distribution on $\mathbb{R}^k$. Consequently, Every Kingman convolution algebra $\big(\mathcal{P}(\mathbb{R}^{+k}), \bigcirc_{s,k}\big)$ is representable in the ordinary convolution algebra $\big(\tilde{\mathcal{P}}_{s,k}(\mathbb{R}^{+k}), \star\big)$ and the map $H_{s,k}$ stands for a homeomorphism.

Proof. First we prove the equation (17) by taking the Fourier transform of the right-hand side of (16). We have, for $\mathbf{t} \in \mathbb{R}^k$,

$$\begin{aligned} \mathcal{F}(H_{s,k}(\mathbf{G}))(\mathbf{t}) &= \int_{\mathbb{R}^k} \Pi_{j=1}^k \cos(t_j x_j) H_{s,k}(\mathbf{G}) d\mathbf{x} \\ &= \int_{\mathbb{R}^k} \int_{\mathbb{R}^{+k}} \Pi_{j=1}^k \cos(t_j x_j)(T_{c_j} F_s(d\mathbf{x})\mathbf{G}(d\mathbf{c}) \\ &= \int_{\mathbb{R}^{+k}} \prod_{j=1}^k \Lambda_s(t_j c_j)\mathbf{G}(d\mathbf{c}) = \hat{\mathbf{G}}(\mathbf{t}) \end{aligned} \tag{20}$$

which implies that the set set $\tilde{\mathcal{P}}_{s,k}(\mathbb{R}^{+k})$ is closed w.r.t. the weak convergence and the ordinary convolution $*$ and, moreover the equations (18) and (19) hold. □

Definition 1. *A p.m. $\mathbf{F} \in \mathcal{P}(\mathbb{R}^{+k})$ is called $\bigcirc_{s,k}-$infinitely divisible ($\bigcirc_{s,k}-ID$), if for every m=1, 2, ... there exists $\mathbf{F}_m \in \mathbf{P}(\mathbb{R}^{+k})$ such that*

$$\mathbf{F} = \mathbf{F}_m \bigcirc_{s,k} \mathbf{F}_m \bigcirc_{s,k} \ldots \bigcirc_{s,k} \mathbf{F}_m \quad (m\ times). \tag{21}$$

Definition 2. $\mathbf{F}$ *is called stable, if for any positive numbers a and b there exists a positive number c such that*

$$T_a\mathbf{F} \bigcirc_{s,k} T_b\mathbf{F} = T_c\mathbf{F} \tag{22}$$

By virtue of Theorem 2 it follows that the following theorem holds true.

Theorem 3. *A p.m. $\mathbf{G}$ is $\bigcirc_{s,k} -ID$, resp. stable if and only if $H_{s,k}(\mathbf{G})$ is ID, resp. stable, in the usual sense.*

The following lemma will be used in the representation of $\bigcirc_{s,k} - ID, k \geq 2$.

Lemma 1. *(i) For every $t \geq 0$*

$$\lim_{x\to 0} \frac{1 - \Lambda_s(tx)}{x^2} = \lim_{x\to 0} \frac{1 - Ee^{it\theta}}{x^2} = \frac{t^2}{2}. \tag{23}$$

(ii) For any $\mathbf{x} = (x_0, x_1, \cdots, x_k)$ and $\mathbf{t} = (t_0, t_1, \cdots, t_k) \in \mathbb{R}^{k+1}, k = 1, 2, ...$

$$lim_{\rho\to 0} \frac{1 - \prod_{r=0}^k \Lambda_s(t_r x_r)}{\rho^2} = \frac{1}{2}\Sigma_{r=0}^k \lambda_r^2(Arg(\mathbf{x}))t_r^2, \tag{24}$$

where $\rho = ||\mathbf{x}||, Arg(\mathbf{x}) = \frac{\mathbf{x}}{||\mathbf{x}||}$, and $\lambda_r(Arg(\mathbf{x})), r = 0, 1, ..., k$ are given by

$$\lambda_r(Arg(\mathbf{x})) = \begin{cases} \cos\phi & r = 0, \\ \sin\phi \sin\phi_1 \cdots \sin\phi_{r-1}\cos\phi_r & 1 \leq r \leq k-2, \\ \sin\phi\sin\phi_1 ... \sin\phi_{k-2}\cos\psi & r = k-1, \\ \sin\phi\sin\phi_1 ... \sin\phi_{k-2}\sin\psi & r = k, \end{cases} \tag{25}$$

where $0 \leq \psi, \phi, \phi_r \leq \pi/2, r = 1, 2, ..., k-2$ are angles of $\mathbf{x}$ appearing its polar form.

The following theorem gives a representation of rad.ch.f.'s of $\bigcirc_{s,k}-$ID distributions (see [13]), Theorem 2.6 for the proof).

Theorem 4. *A p.m. $\mu \in ID(\bigcirc_{s,k})$ if and only if there exist a σ-finite measure M (a Lévy's measure) on $\mathbb{R}^{+k}$ with the property that $M(\mathbf{0}) = 0, \mathbf{M}$ is finite outside every neighborhood of $\mathbf{0}$ and*

$$\int_{\mathbb{R}^{+k}} \frac{\|\mathbf{x}\|^2}{1+\|\mathbf{x}\|^2}\mathbf{M}(d\mathbf{x}) < \infty \tag{26}$$

and for each $\mathbf{t} = (t_1, ..., t_k) \in \mathbb{R}^{+k}$

$$-\log \hat{\mu}(\mathbf{t}) = \int_{\mathbb{R}^{+k}} \{1 - \prod_{j=1}^{k} \Lambda_s(t_j x_j)\} \frac{1+\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2} M(\mathbf{dx}), \tag{27}$$

where, at the origin $\mathbf{0}$, the integrand on the right-hand side of (27) is assumed to be

$$\Sigma_{j=1}^{k} \lambda_j^2 t_j^2 = lim_{\|\mathbf{x}\| \to 0} \{1 - \prod_{j=1}^{k} \Lambda_s(t_j x_j)\} \frac{1+\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2} \tag{28}$$

for nonnegative $\lambda_j, j = 1, 2, ..., k$ given by equations (25) in Lemma 1. In particular, if $M = 0$, then μ becomes a Rayleighian distribution with the rad.ch.f (see definition 3)

$$-\log \hat{\mu}(\mathbf{t}) = \frac{1}{2} \sum_{j=1}^{k} \lambda_j^2 t_j^2, \quad \mathbf{t} \in \mathbb{R}^{+k}, \tag{29}$$

for some nonnegative $\lambda_j, j = 1, ..., k$.
Moreover, the representation (27) is unique.

An immediate consequence of the above theorem is the following:

Corollary 1. *Each distribution $\mu \in ID(\bigcirc_{s,k})$ is uniquely determined by the pair $[\mathbf{M}, \boldsymbol{\lambda}]$, where $\mathbf{M}$ is a Levy's measure on $\mathbb{R}^{+k}$ such that $\mathbf{M}(\mathbf{0}) = 0$, $\mathbf{M}$ is finite outsite every neighbourhood of $\mathbf{0}$ and the condition (26) is satisfied and $\boldsymbol{\lambda} := \{\lambda_1, \lambda_2, \cdots \lambda_k\} \in \mathbb{R}^{+k}$ is a vector of nonnegative numbers appearing in (29). Consequently, one can write $\mu \equiv [\mathbf{M}, \boldsymbol{\lambda}]$.*

In particular, if $\mathbf{M}$ is zero measure then $\mu = [\boldsymbol{\lambda}]$ becomes a Rayleighian p.m. on $\mathbb{R}^{+k}$ as defined as follows:

Definition 3. *A k-dimensional distribution, say $\boldsymbol{\Sigma}_{s,k}$, is called a Rayleigh distribution, if*

$$\boldsymbol{\Sigma}_{s,k} = \sigma_s \times \sigma_s \times \cdots \times \sigma_s \quad (k\ times). \tag{30}$$

Further, a distribution $\mathbf{F} \in \mathcal{P}(\mathbb{R}^{+k})$ is called a Rayleighian distribution if there exist nonnegative numbers $\lambda_r, r = 1, 2 \cdots k$ such that

$$\mathbf{F} = \{T_{\lambda_1}\sigma_s\} \times \{T_{\lambda_2}\sigma_s\} \times \ldots \times \{T_{\lambda_k}\sigma_s\}. \tag{31}$$

It is evident that every Rayleigh distribution is a Rayleighian distribution. Moreover, every Rayleighian distribution is $\bigcirc_{s,k}-$ID. By virtue of (7) and (30) it follows that the k-dimensional Rayleigh density is given by

$$g(\mathbf{x}) = \Pi_{j=1}^{k} \frac{2^k (s+1)^{k(s+1)}}{\Gamma^k(s+1)} x_j^{2s+1} exp\{-(s+1)||\mathbf{x}||^2\}, \tag{32}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^{+k}$ and the corresponding rad.ch.f. is given by

$$\hat{\Sigma}_{s,k}(\mathbf{t}) = Exp(-|\mathbf{t}|^2/4(s+1)), \quad \mathbf{t} \in \mathbb{R}^{+k}. \tag{33}$$

Finally, the rad.ch.f. of a Rayleighian distribution $\mathbf{F}$ on $\mathbb{R}^{+k}$ is given by

$$\hat{\mathbf{F}}(\mathbf{t}) = Exp(-\frac{1}{2}\sum_{j=1}^{k} \lambda_j^2 t_j^2) \tag{34}$$

where $\lambda_j, j = 1, 2, \ldots, k$ are some nonnegative numbers.

3. An analogue of the Lévy-Cramér Theorem in k-dimensional Kingman convolution algebras

We say that a distribution $\mathbf{F}$ on $\mathbb{R}^k$ has dimension m, $1 \leq m \leq k$, if m is the dimension of the smallest hyper-plane which contains the support of $\mathbf{F}$. The following theorem can be regarded as a version of the Lévy-Cramér Theorem for multi-dimensional Kingman convolution.The case k=1 was proved by Urbanik ([18]).

Theorem 5. *Suppose that $\mathbf{G}_i \in \mathcal{P}(\mathbb{R}^{+k}), i = 1, 2$ and*

$$\Sigma_{s,k} = \mathbf{G} \bigcirc_{s,k} \mathbf{K}. \tag{35}$$

Then, $\mathbf{G}$ and $\mathbf{K}$ are both Rayleighian distributions fufilling the condition that there exist nonnegative numbers λ_r and $\gamma_r, r = 1, 2, \ldots, k$ such that the number of non-zero coefficients $\lambda_r' s$ and $\gamma_r' s$ are equal to the dimensions of $\mathbf{G}$ and $\mathbf{K}$, respectively, and moreover,

$$\lambda_r^2 + \gamma_r^2 = 1, \qquad r = 1, 2, ..., k \tag{36}$$

and

$$\mathbf{G} = T_{\lambda_1}\sigma_s \times T_{\lambda_2}\sigma_s \times \ldots \times T_{\lambda_k}\sigma_s \tag{37}$$

and

$$\mathbf{K} = T_{\gamma_1}\sigma_s \times T_{\gamma_2}\sigma_s \times \ldots \times T_{\gamma_k}\sigma_s. \tag{38}$$

Proof. Suppose that the equation (35) holds. Using the map $H_{s,k}$ we have

$$H_{s,k}(\Sigma_{s,k}) = H_{s,k}(\mathbf{G}) * H_{s,k}(\mathbf{K})$$

which implies that

$$N(\mathbf{0}, \mathbf{I}) = H_{s,k}(\mathbf{G}) * H_{s,k}(\mathbf{K}).$$

By the well-known Lévy-Cramér Theorem on $\mathbb{R}^k$ (cf. Linnik and Ostrovskii [9]), that they are both symmetric Gaussian distributions on $\mathbb{R}^k$. Consequently, they must be of the form (37) and (38) and the coefficients $\lambda_r' s$ and $\gamma_r' s$ satisfy the above stated conditions. □

4. Remarks on $\mathbb{R}^k$.-valued Bessel processes

Since every Rayleigh distribution $\Sigma_{s,k}$ is ID the semigroup $\Sigma_{s,k}^t, t \geq 0$, where the power is taken in the sense of the Kingman convolution $\bigcirc_{s,k}$, induces a homogeneous Markov process $\mathbf{B}_t, t \geq 0$ which is the Bessel process fulfiling the condition that

$$\Sigma_{s,k} \stackrel{d}{=} \mathbf{B}_1. \tag{39}$$

By virtue of (17) it follows that every Bessel process is an ordinary symmetric Lévy process which together with the fact that the sample functions of the Bessel process are continuous with (P.1) implies that it is a Brownian motion on $\mathbb{R}^{+k}$.

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